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TITLE "PHASE DYNAMICS - A REVIEW AND A PERSPECTIVE"

AUTHOR(S) HELMUT R. BRAND

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Los Alamos

Los Alamos National Laboratory
Los Alamos, New Mexico 87545

PHASE DYNAMICS - A REVIEW AND A PERSPECTIVE

Helmut R. Brand

FB 7, Physik, Universität Essen
D43 Essen 1, W. Germany

and

Center for Nonlinear Studies, MS - B258
Los Alamos National Laboratory, University of California
Los Alamos, NM 87545, U.S.A.

We critically review the present status of phase dynamics, the analogue of hydrodynamics for large aspect-ratio pattern forming non-equilibrium systems. Low frequency, long wavelength excitations in systems such as the regular patterns which arise in the Taylor instability (a simple fluid in the gap between two concentric cylinders is subjected to a torque by rotating the inner or both cylinders) or the Benard instability (a thin layer of fluid is heated from below) are examined and the question of nonlinear excitations such as solitary waves is addressed. The connection between the phenomenological parameters arising in the phase equations and results obtained from amplitude equations valid close to the onset of a specific pattern are investigated. Suggestions for experiments to test further the concept of phase dynamics are included.

Introduction

That there is a strong similarity between phase transitions in equilibrium systems (as e.g. the paramagnetic-ferromagnetic transition in magnetic systems or the superfluid-normal fluid transition in ^4He) and nonequilibrium phase transitions is well established since the pioneering work of Graham and Haken [1] and deGiorgio and Scully [2], who pointed out the analogy between second order phase transitions and the onset of laser action in the single mode laser in the field of quantum optics. Since then many examples of such a similarity have been found in nonequilibrium systems including fields such as quantum optics, hydrodynamic instabilities, autocatalytic chemical reactions, nonlinear oscillators in diverse fields etc.

After this analogy has been well established it seems natural to ask whether there is also a non-equilibrium analogue for the theory close to thermodynamic equilibrium, which deals with the low frequency, long wavelength excitations around a given ground state, namely for hydrodynamics. This question seems to be all the more important, since a considerable body of work has been accumulated over the last few years dealing with the influence of spatial degrees of freedom on nonequilibrium systems, especially in connection with hydrodynamic instabilities [3-12] .

In the present contribution we propose that such an approach indeed exists: phase dynamics, the analogue of hydrodynamics for large aspect ratio pattern forming nonequilibrium systems. The role played by atoms or molecules in the derivation of hydrodynamic equations is taken over by vortices, rolls or hexagons for pattern forming nonequilibrium systems. This fact implies that such an approach can only be applicable if there is a sufficiently high number of unit cells (rolls etc.) in the container. It turns out that this number is of order ten or so, but it might be sometimes even lower. As for the case of hydrodynamics for equilibrium systems, phase dynamics can only be applicable for ground states with a well defined symmetry. I.e. one has to assume e.g. that one is in a given state such as the Taylor vortex flow in the Taylor instability. The symmetries of this nonequilibrium ground state are then used to determine the macroscopic variables, which govern the long wavelength, low frequency excitations of the ground state under consideration. We will also discuss, how the phenomenological coefficients appearing in the phase equations can be evaluated approximately from a more microscopic approach such as amplitude equations, which are valid close to the onset of an instability (and which are the analog of a Ginzburg-Landau approach for the order parameter in systems close to thermodynamic equilibrium).

The key question in this connection which clearly needs further experimental and theoretical studies, is:

Can phase dynamics serve as a next step in the hierarchy Newton's law, Boltzmann equations or dynamic equation of motion for the Green's function,hydrodynamics,

a sequence that is obtained by investigating the dynamic behavior on larger and larger length scales after averaging out the information contained in higher wave vectors.

In this contribution we will exploit the gateway of attack outlined above to investigate the long wavelength, low frequency behaviour of a number of systems such as the spatially periodic or multiply periodic states arising in the Taylor and Benard instability.

Questions addressed include the following. Which analogues of broken symmetries (such as gauge invariance, broken translational symmetry etc.) have been established in large aspect ratio pattern forming nonequilibrium systems? What are the hydrodynamic excitations, i.e. is there an analogue of diffusive and propagating modes? Do the conserved quantities of hydrodynamics such as density of linear momentum have an analogue in phase dynamics? Can defects be incorporated into the phase dynamic description? Does phase dynamics help to understand pattern selection problems? Are there critical experiments, which can test the predictions made? Is it possible that we learn something for the onset of turbulence from phase dynamics, if the motion of defects is included in the description?

Phase diffusion

The first step to reach the goal outlined in the introduction was done by Pomeau and Manneville [13] in 1979. They investigated the dynamics of slow spatial modulations for a model of convective rolls, the Swift-Hohenberg equation [14]

$$A_t = \varepsilon A - (\partial_{xx} + \partial_{yy} + q_0^2)^2 A - A^3 \qquad (1.1)$$

where ∂_{xx} denotes the second derivative in x-direction in the xy plane of the convective layer, ε measures the relative distance from onset and where q_0 is the critical wave vector at onset.
Using the gradient as an expansion parameter, Pomeau and Manneville derived, using a reduced perturbation expansion, a dynamic equation for the slow spatial changes of a spatially periodic solution of (1.1), which can be expressed in terms of the spatial and temporal variations of the phase ϕ associated with the position of the rolls As a result the equation

$$\phi_t = D_1 \phi_{xx} + D_2 \phi_{yy} \qquad (1.2)$$

emerges, where x denotes the direction parallel to the normal of the convective rolls and y the direction along the crest of the rolls. Since the starting point was a concrete microscopic model, namely the Swift-Hohenberg equation, explicit expressions for D_1 and D_2 have been obtained in [13] as functions of the parameters contained in (1.1). Those can in turn be determined approximately close to the onset of the instability from the coefficients entering the Navier Stokes equations such as kinematic viscosity etc.. This situation is reminiscent of the same problem that arises in hydrodynamics close to thermodynamic equilibrium: to determine quantities such as the thermal diffusivity or the thermal expansion coefficient, which arise as phenomenological coefficients when deriving the hydrodynamic equations, one has to resort to more microscopic techniques as e.g. the Boltzmann equation. Or one has to determine the corresponding parameters experimentally.

The structure of (1.1) can be obtained by a simple alternative consideration. Taking into account that the ground state (the set of regular, parallel rolls) satisfies $x \to -x$ and $y \to -y$ symmetry and keeping in mind the fact that the variation of the phase is the only candidate for a macroscopic variable compatible with all the symmetries of the ground state considered, one arrives immediately at (1.2).

The validity of (1.2) has been investigated in detail experimentally by Wesfreid and Croquette [15,16] and complete agreement was found. It seems important to note that (1.2) is contained implicitly in the Newell-Whitehead-Segel equation [17,18], the amplitude equation which is applicable close to the onset of convection in a simple fluid in the high Prandtl number limit. Based on a similar amplitude equation approach, Ortoleva and Ross have derived a phase equation of the form (1.2) in the context of autocatalytic chemical reactions [19]. A generalization of (1.2) to cross-rolls was considered by Zaleski [20] and the existence of two coupled, purely diffusive modes emerged from his study. Another application results for the phase diffusion in the Taylor vortex flow of the Taylor instability and an equation isomorphic to (1.2) applies in this case [21,22]. The importance of the diffusion of a perturbation in the vortex diameter for the vortex flow was recognized first by Snyder [23], although the terminology and the framework of looking at the problems had not been developed at that time.

In closing the section on phase diffusion we briefly outline the generalization of (1.2) into the nonlinear domain, which has been given in [24]. Taking into account higher order derivatives the equation

$$\psi_t = D_1 \psi_{xx} + D_1' \psi_{xxxx} + E_1 \psi_{xx} \psi_x \qquad (1.3)$$

$$= (D_1 + D_1' \partial_{xx} + E_1 \psi_x) \psi_{xx}$$

obtains, if only spatial variations along the axis of the cylinders are considered. (1.3) allows for static solutions in the form of a hyperbolic tangens, but there is probably no physical significance associated with these solutions, since the 'potential', which can be associated with (1.3), is not bounded from below [24]. If higher order nonlinearities are included, the resulting equation is considerabiy more complicated than (1.3) and neither a static solution nor a potential has been found in this case [24]. An equation isomorphic to (1.3) can be equally well derived for the nonlinear variations of the phase for the roll state of the Rayleigh Benard instability and for any large aspect ratio pattern forming nonequilibrium system, which has only one phase and the same symmetry as the Taylor vortex flow.

Phase dynamics for the wavy mode and the modulated wavy mode in the Taylor instability

In the Taylor Couette instability the flow between two concentric cylinders with the outer one fixed (or rotating) and the inner one rotating is investigated. The first instability observed for fixed outer cylinder is the Taylor vortex flow, in which vortex pairs with their axis parallel to the cylindrical axis emerge. At higher rotation rates one observes the Taylor wavy mode state, which has no longer the azimuthal symmetry of the Taylor vortex flow, since periodic displacements in the vortices propagate around the cylinders. The wavy mode flow arises for cylinders with a small gap at Taylor numbers which are only slightly higher than those for the onset of vortices.

The description of long wavelength, low frequency perturbations for the vortex flow and the wavy mode state has been given in [21,24] . To describe the wavy mode state two phases are necessary. One (ϕ) gives the azimuthal position of the waves and the other one (ψ) characterizes the position of the vortices along the cylindrical axis. In both cases the phases are associated with changes in position, as is the case for the displacement vector in solids. I.e. we have, just as in the case of Rayleigh Benard convection discussed in the last chapter, the analogues of variables characterizing broken translational symmetries. To derive phase dynamic equations we must keep in mind the symmetries of the nonequilibrium ground state under consideration. First we have, as for Rayleigh Benard convection, the symmetry $x \to -x, \psi \to -\psi$ where x denotes the coordinate along the cylindrical axis. In addition the wavy mode state has, due to the presence of the waves, the symmetry $y \to -y, \psi \to \psi$, $\phi \to -\phi$, where y is the azimuthal direction. For the linearized phase equations we find

$$\psi_t = D_1 \psi_{xx} + C_1(q_y)\, \phi_x \qquad (2.1)$$

$$\phi_t = D_2 \phi_{xx} + C_2(q_y)\, \psi_x \qquad (2.2)$$

In writing down (2.1),(2.2) we have ignored spatial inhomogeneities in azimuthal direction and we have denoted the wave vector in this direction as q_y. The dependence of the cross couplings in (2.1),(2.2) on q_y has been emphasized since the symmetry of the wavy mode state in azimuthal direction requires C_1 and C_2 to be odd functions of q_y. The additional assumption of analyticity then implies for small values of q_y : $C_1, C_2 \propto q_y$. In (2.1),(2.2) ϕ_t gives the change in frequency of the wavy motion and ψ_x the change in the wave number of the vortices.

As is readily shown [21], the static solution of (2.1),(2.2) can account for the experimental observation [8], that in the wavy mode state the wavelength of the vortices varies over a long length scale, which is in contrast to the case of rolls in the Benard instability and the vortices in the Taylor vortex state, in which the wavelength is essentially constant over the bulk of the cell.

To investigate the dynamic consequences of (2.1),(2.2) we study the time evolution of small perturbations of the wave number. Using a disturbance $\propto \exp[i(kx-\omega t)]$ we find a propagating mode for $k \to 0$ with frequency $\omega = \pm (C_1C_2)^{1/2} k$ and damping $\propto k^2$. For larger k we obtain

$$\omega = - i/2\, (D_1 + D_2)\, k^2 + 1/2\, k\, [\, 4\, C_1C_2 - k^2\, (D_1 - D_2)^2]^{1/2} \qquad (2.3)$$

for $4\, C_1C_2 > k^2\, (D_1 \;\; D_2)^2$ and

$$\omega = - i(1/2\, (D_1 + D_2)\, k^2 \pm 1/2\, k\, [\, -4\, C_1C_2 + k^2\, (D_1 - D_2)^2]^{1/2}\,) \qquad (2.4)$$

for $4\, C_1 C_2 < k^2\,(D_1 - D_2)^2$. Thus we [21] find with increasing wave vector first a propagating mode, then an overdamped mode and finally a purely diffusive mode. In writing down (2.3),(2.4) C_1 and C_2 have been assumed to be positive as it also follows from the amplitude equation in the small gap limit given below. A change in sign of either C_1 or C_2 would signal the onset of an additional instability.

In retrospect it seems worthwhile to mention that the wavy mode state of the Taylor instability has been the first nonequilibrium ground state for which a propagating mode in phase dynamics has been predicted.

To incorporate nonlinearities and higher order gradient terms in the phase equations we proceed as above for the derivation of the nonlinear diffusion equation for Benard rolls in convection. Taking into account the symmetries of the ground state of the Taylor wavy mode we obtain in lowest order in the nonlinearities [24]

$$\psi_t = D_1\, \psi_{xx} + C_1(q_y)\, \phi_x + D'_1\, \psi_{xxxx} + C'_1(q_y)\, \phi_{xxx} \qquad (2.5)$$

$$+ E_1\, \psi_{xx}\, \psi_x + E_2\, \psi_x\, \phi_x + E_3\, \phi_{xx}\, \phi_x$$

$$\phi_t = D_2\, \phi_{xx} + C_2(q_y)\, \psi_x + D'_2\, \phi_{xxxx} + C'_2(q_y)\, \psi_{xxx} \qquad (2.6)$$

$$+ F_1\, \phi_x^2 + F_2\, \psi_x^2 + F_3\, \phi_{xx}\, \psi_x + F_4\, \phi_x\, \psi_{xx}$$

For a detailed discussion of (2.5) and (2.6) we refer to [24]. We would like to draw the attention of the reader to one quite interesting special case of these two equations: if one is able to keep the wavelength of the vortices fixed experimentally, the wavy mode state of the Taylor instability could serve as a testing ground of the Kuramoto Sivashinsky equation [25,26]

$$\phi_t = D_2\, \phi_{xx} + D'_2\, \phi_{xxxx} + F_1\, \phi_x^2 \qquad (2.7)$$

as a phase equation. To clamp the vortex wavelength in the wavy mode state would thus be worthwhile, since no experimental system satisfying (2.7) has been reported so far (cf. also our discussions on the importance of Galilean invariance in one of the sections below).

To evaluate the coefficients appearing in the phase equations one can use, as discussed above, an amplitude equation valid close to onset of the instability. Using the technique of Newell and coll.[17, 27-31] one obtains [21] in the small gap limit

$$A_t = \tau_0^{-1} \left(A + (\xi_x^2 + \xi_y^2) A - g\, |A|^2 A \right) + i\, T_c^{1/2} s_1 A_{xy} \tag{2.8}$$

where A(x,y,t) is the complex envelope function and where we have used appropriately scaled variables (cf. [21] for details). From this amplitude equation it is possible to extract values for all coefficients in (2.1),(2.2) close to onset [21] by treating the wavy mode state as a superposition of an amplitude for the vortices and one for the waves, which vanishes at the wavy mode onset. This amplitude equation approach has been supplemented by studies of the effect of finite length of the cylinders [32] and by investigations of the influence of pressure variations [33]. Expressions for the nonlinear coefficients can be easily evaluated close to onset using a technique devised by Kuramoto [34].

In the modulated wavy mode state, which has been observed at higher rotation frequencies than the wavy mode state in some cases [35], one has two waves propagating in azimuthal direction. In [24] we have proposed a phase dynamic description of this state using two phases ϕ and ω to describe the travelling waves. As a result we obtain three linearized phase equations, which give rise to one pair of propagating modes and one purely diffusive mode in the limit of small wave numbers. That one can have simultaneously propagating modes and diffusive modes in linearized phase dynamics compares well with hydrodynamics in systems close to thermodynamic equilibrium, where one has e.g. in a simple fluid sound waves and heat and vorticity diffusion. The details of the phase dynamics for the modulated wavy mode state can be found in [24].
In closing this section we remark that we have used two different types of phases to characterize the wavy mode state and the modulated wavy mode state of the Taylor instability: one type is even under $x \to -x$ and the other one is

odd. This feature is reminiscent of the different behaviour of hydrodynamic variables under spatial parity. The simultaneous presence of both types allowed for the possibility of a propagating mode and we have seen that the nonlinearities in the phase equations (2.5),(2.6) also strongly reflect the different symmetry of the phases involved.

Phase dynamics for spiraling Taylor vortices and for interpenetrating spirals

In addition to the Taylor vortex flow, the wavy mode and the modulated wavy mode state, which are periodic along the axis of the cylinders, there are also two helical flow states for counter-rotating cylinders: the spiraling Taylor vortices, which are characterized by one helix propagating parallel to the axis of the cylinders [3,4] and the interpenetrating spiral state, in which two interwoven helices, which are probably incommensurate, propagate along the axis of the cylinders [4]. Single propagating helices have also been observed in configurations with a through-flux and only the inner cylinder rotating with the outer one at rest [36,37] and for the case where both cylinders are at rest [38]. In the latter case the gap between the two cylinders was filled with mercury, a magnetic field was applied parallel to the cylindrical axis and an electric current in radial direction. The action of the Lorentz force then makes the occurence of a spiral intuitively quite plausible.

A phase dynamic description of both, the spiraling Taylor vortices and the interpenetrating spirals, has been given in references [39,40] and we will therefore focus on the description of the main ingredients and the results.
For the spiraling vortices the phase changes in a helicoidal way according to

$$\phi = 2\,\pi/\lambda\, z + m\,\Theta + \phi_0 \qquad (3.1)$$

and the ground state is, in addition, characterized by a frequency ω_0. In (3.1) λ is the wavelength of the periodic structure, z has been chosen to be parallel to the cylindrical axis, Θ is the azimuthal coordinate, m an integer and ϕ_0 a constant. In the experiments values of m between 1 and 4 have been observed. If m is e.g. equals to 3, this means that going up along the z-axis by one unit is equivalent to a change of Θ by 120°.

The macroscopic helical structure in the nonequilibrium system discussed here is reminiscent of two equilibrium system showing also a macroscopic helical structure: cholesteric and chiral smectic liquid crystals [41]. As it is known e.g. from the optical [41] and hydrodynamic [42-44] behaviour of these phases, the loss of mirror symmetry is strongly reflected in the macroscopic properties of these materials. Just as for chiral smectics and cholesterics one can associate a pseudoscalar quantity with the spiraling Taylor vortices: the helical wavevector q_0 of the nonequilibrium structure.
If we denote the only phase variable associated with the spiraling vortices by ϕ, we obtain from general symmetry arguments the linearized phase equation

$$\phi_t = C\, q_0 \cdot \partial_z \phi + D_\perp\, \partial_{\perp\perp} \phi + D_z\, \partial_{zz} \phi \qquad (3.2)$$

where $D_\perp$ and D_z are the diffusion coefficients perpendicular and parallel to to the axes of the cylinders and the helix. The term proportional to C on the right hand side of (3.2) is directly related to the existence of a pseudoscalar quantity associated with the spiral structure in the present system.
To see the physical implications of (3.2) we study plane wave perturbations and obtain for these from (3.2) the dispersion relation

$$\omega = - k_z\, C\, q_0 + i\,(D_\perp k_z^2 + D_z\, k_\perp^2) \qquad (3.3)$$

By inspection of (3.3) we see immediately that perturbations parallel to the axis of the helix can propagate with velocity Cq_0. Only fluctuations strictly perpendicular to the helical axis show purely dissipative behavior. Interestingly enough spiraling Taylor vortices can show a propagating mode with only one macroscopic variable. Such a phenomenon is unknown from hydrodynamic systems close to local thermodynamic equilibrium. Previously at least two variables (phase or hydrodynamic) were thought to be necessary for the occurence of a propagating mode. The existence of this novel and unique behavior in condensed matter physics can be traced back to the fact that the nonequilibrium ground state of the present system breaks time reversal symmetry and violates spatial parity. It also seems important to stress that the broken symmetry associated with the existence of the macroscopic helix is a combined translational - rotational symmetry.

The nonlinear phase dynamic equation given in [39] also reflects the loss of mirror symmetry and contains - even in lowest order in the nonlinearities - a large number of terms.

The simplest amplitude equation, which can be written down using the technique of Newell and coll. [17,27-31], takes the form [39]

$$A_t = \varepsilon A - \beta |A|^2 A + D_1 \partial_{\perp\perp} A + D_2 \partial_{zz} A + \gamma q_0 \partial_z A \qquad (3.4)$$

Close to threshold $C \propto \gamma$. An amplitude equation of the form (3.4) has also been studied, along with more complex models, in [45]. For $q_0 = 0$, (3.2) and (3.4) reduce to the corresponding equations for the Taylor vortex flow discussed in the last section.

Interpenetrating spirals have so far only been observed for counter-rotating cylinders [4]. In [40] we have suggested for the first time, triggered by the observations described in [4], that interpenetrating spirals represent an incommensurate nonequilibrium ground state. This implies that they can be described by two phases ϕ_1 and ϕ_2 as follows

$$\phi_1 = 2\pi/\lambda_1 z + m\Theta + \phi_{10} \qquad (3.5)$$

$$\phi_2 = 2\pi/\lambda_2 z + n\Theta + \phi_{20} \qquad (3.6)$$

where λ_1 and λ_2 are the incommensurate wavelengths of the two helices, n and m are integers, ϕ_{10} and ϕ_{20} are constants and where Θ is as above the azimuthal coordinate. We also note, that in contrast to the single spiral state, interpenetrating spirals have no analogue in systems close to thermodynamic equilibrium.

Since one has, by assumption, two incommensurate spirals, there are also two pseudoscalar quantities q_1 and q_2. One spiral can be moved in this case by an infinitesimal amount with respect to the other without restoring force in the long wavelength limit by definition of the term incommensurate. Neglecting the interaction between the two spirals, we obtain, due to the uniaxial overall symmetry, the two equations

$$\phi_{1t} = C_1 q_1 \partial_z\phi_1 + (D_{\perp 1}\partial_{\perp\perp} + D_{z1}\partial_{zz})\phi_1 + (d_{\perp 2}\partial_{\perp\perp} + d_{z2}\partial_{zz})\phi_2 \qquad (3.7)$$

$$\phi_{2t} = C_2 q_2 \partial_z\phi_2 + (D_{\perp 2}\partial_{\perp\perp} + D_{z2}\partial_{zz})\phi_2 + (d_{\perp 1}\partial_{\perp\perp} + d_{z1}\partial_{zz})\phi_1 \qquad (3.8)$$

There is no term proportional to $\partial_z\phi_2$ in (3.7) to preserve incommensurability and vice versa for (3.8).

To incorporate the effect of a finite interaction between the spirals one can proceed in the same way as in the case of hydrodynamics for incommensurate equilibrium systems [46]. We then find [40], that the sum of the two phases is still strictly hydrodynamic, whereas the difference of the two relaxes in a long, but finite time thus giving rise to a hydrodynamic excitation with a gap, which turns out to be proportional to the strength of the interaction.

I. e. as the wavenumber of the investigated perturbation is reduced, we predict a cross-over from two propagating modes to one, with the latter being accompanied by a mode with a gap at k=0. Equations (3.7) and (3.8) can be easily obtained near threshold from the two amplitude equations

$$A_t = \varepsilon_1 A - \beta_1 |A|^2 A - \delta_1 |B|^2 A + D_1 \partial_{\perp\perp} A + D_2 \partial_{zz} A$$

$$+ D_3 \partial_{\perp\perp} B + D_4 \partial_{zz} B + \gamma_1 q_1 \partial_z A \qquad (3.9)$$

$$B_t = \varepsilon_2 B - \beta_2 |B|^2 B - \delta_2 |A|^2 B + D_5 \partial_{\perp\perp} B + D_6 \partial_{zz} B$$

$$+ D_7 \partial_{\perp\perp} A + D_8 \partial_{zz} A + \gamma_2 q_2 \partial_z B \qquad (3.10)$$

Equations (3.9) and (3.10) are applicable to the case of non-interacting interpenetrating spirals. As it is easily checked, a finite interaction changes the structure of the amplitude equations so as to give rise to the microscopic mode for the difference of the phases in phase dynamics.
To describe the case of incommensurability of a stationary roll pattern, as it has been observed for the spatially modulated electro-hydrodynamic instability in nematic liquid crystals [5,47], the approach outlined above just goes through in parallel. All one has to do is to put $C_1 = C_2 = 0$. I.e. in this case we predict a cross-over from two diffusive modes to one microscopic and one diffusive mode.
Amplitude equations for the case of spatially periodic forcing have been discussed for this case by Coullet et al. [48,49] and we refer the reader to this work for the details.
In closing this section we would like to point out, that similar considerations as the ones presented here for incommensurate systems go through equally well [40] for the nonequilibrium analogues of icosahedral phases [50,51], in case those should ever be found in nature.
In conclusion it turns out that incommensurate nonequilibrium systems bring along additional phase variables, but they are most likely not truly hydrodynamic ($\omega \rightarrow 0$ for $k \rightarrow 0$), just as it has also been the case for the phasons in incommensurate systems close to thermodynamic equilibrium.

Phase dynamics in the vicinity of a co-dimension two point

After we have discussed above several examples for the appearance of propagating modes in phase dynamics, mainly in connection with the spatio-temporal structures in the Taylor instability, we investigate now the phase dynamics in the vicinity of a multicritical point, namely a co-dimension two point with double eigenvalue zero. That a propagating mode might be possible in this case can be inferred immediately from the amplitude equation including slow spatial modulations in the direction parallel to the layer normal. This equation reads [52]

$$w_{tt} - \{ \alpha \, (\varepsilon_0 + \xi_0^2 \, \partial_{xx}) - f_2 \, | w |^2 \} \, w_t$$

$$- \{ \beta \, (\varepsilon_s + \xi_s^2 \, \partial_{xx}) + f_1 | w |^2 \} \, w = 0 \quad (4.1)$$

Inserting the ansatz $w = r \cdot \exp(i\phi)$ in (4.1), a linearised phase equation of the form

$$\phi_{tt} + a \, \phi_t + b \, \phi_{xx} + c \, \phi_{xxt} = 0 \qquad (4.2)$$

with $a = - \alpha \, \varepsilon_0$, $b = - \beta \, \xi_s^2$, $c = - \alpha \, \xi_0^2$ results.

Eq.(4.2) is associated with the amplitude equation (4.1) in the same way as the phase diffusion equation (1.2) is related to the Newell-Whitehead-Segel equation [17,18].

Looking for plane wave solutions we obtain from (4.2) after a fourier transform in time and space the dispersion relations:

a) for $a \neq 0$ and arbitrary, but macroscopically small k two over-damped modes

$$\omega_{1,2} = 0.5 \cdot (i(a + c \, k^2) \pm [-4 \, b \, k^2 - (a + c \, k^2)^2]^{1/2}) \qquad (4.3)$$

b) for $a \neq 0$ and $k \to 0$

$$\omega_{1,2} = ia \pm |b|^{1/2} k + O(ik^2) \tag{4.4}$$

c) for $a = 0$, i.e. at the onset of the oscillatory instability, purely propagative behavior

$$\omega_{1,2} = \pm |b|^{1/2} k + i/2\, c\, k^2 \tag{4.5}$$

I.e. the situation is different from that in incommensurate systems. In those one variable always stays hydrodynamic (no gap in the excitation spectrum for k=0), whereas in the present case both modes - and not just one as in the case of the incommensuate systems for small k - are overdamped.
As all phase dynamic equations, (4.2) is also valid well above the onset of the instabilities, as long as one is in the vicinity of the polycritical point in parameter space. Near onset, however, the coefficients in (4.2) can be determined from the coefficients in the amplitude equation, as listed after (4.2) above.
It is now straightforward to generalize (4.2) into the nonlinear regime using symmetry considerations. The arguments are similar to those presented in the last section and we obtain in the vicinity of a co-dimension two point with double eigenvalue zero the nonlinear phase equation [53]

$$\begin{aligned}\phi_{tt} + a\,\phi_t + b\,\phi_{xx} + c\,\phi_{xxt} + b\,\phi_{xxxx} + c\,\phi_{xxxxt}& \\ + d_1\,\phi_x\,\phi_{xx} + d_2\,\phi_{xx}\,\phi_{xt} + d_3\,\phi_{xxt}\,\phi_x = 0&\end{aligned} \tag{4.6}$$

Equation (4.6) has two obvious applications . First of all it applies near the onset of convection in a binary mixture of miscible fluids (such as e.g. ethanol-water or $^3He/^4He$ mixtures) with or without a porous medium. Secondly it should be also applicable in the vicinity of the co-dimension two point near the onset of convection in nematic liquid crystals. Additional applications can be expected for all systems for which an amplitude equation of the form (4.1) in the

vicinity of a co-dimension two point with double eigenvalue zero can be derived.
The generalization to the case of a co-dimension three bifurcation with a triple zero eigenvalue is straightforward now and we obtain the linearized phase equation

$$\phi_{ttt} + a_1\, \phi_{tt} + a_2\, \phi_t + b_1\, \phi_{xx} + b_2\, \phi_{xxt} + b_3\, \phi_{xxtt} = 0 \qquad (4.7)$$

The discussion of the dispersion relation is similar as for the co-dimension two point with double eigenvalue zero and we obtain also for the co-dimension three point with threefold eigenvalue zero the possibility of a propagating mode in phase dynamics. As above, (4.7) can be easily generalized to the nonlinear domain.

On the importance of Galilean invariance for phase dynamics

Recently several authors (cf. especially Coullet and Fauve [54,55] and Shraiman [56]) have examined the impact of Galilean invariance on phase dynamics and it has been proposed [55], that it leads quite generally to an additional dynamic degree of freedom.
To discuss this question one can follow various paths of thought.
It is e.g. immediately clear that the Newell-Whitehead Segel equation [17,18] does not reflect Galilean invariance.

This is different for the Sivashinsky equation

$$u_t + u_{xx} + u_{xxxx} + uu_x = 0 \qquad (5.1)$$

where u is the amplitude. Equation (5.1) as an amplitude equation has been derived for a number of systems (including flames, combustion and convection between insulating boundaries) by Sivashinsky and collaborators [25,57,58] . Shraiman uses the fact that (5.1) is

Galilean invariant, to derive two nonlinear equations for a phase ϕ and a variable ξ, which corresponds to a weakly inhomogeneous Galilei transformation [56]

$$\phi_t + \xi + \xi\,\phi_x + \gamma\,\xi_{xx} + \alpha\,\phi_{xx} + \beta_\alpha\,\phi_x\,\phi_{xx} = 0 \qquad (5.2)$$

$$\xi_t + \xi\,\xi_x + \nu\,\xi_{xx} + \sigma\,\phi_{xx} + \beta_\sigma\,\phi_x\,\phi_{xx} = 0 \qquad (5.3)$$

It seems important to keep in mind, however, under which assumptions (5.1) was derived for specific physical systems. One always assumes [25,57,58], that the diameter d of the unit cell of the emerging spatial structure is very large compared to all characteristic lengths of the system. Then one performs an expansion in inverse powers of d and obtains in lowest order as a resultat (5.1). I.e. the assumption, upon which the derivation of (5.1) is based, is, that one has only one roll (e.g. for the case of convection between thermally insulating boundaries) in the container. If one takes into account the next order in 1/d, one obtains in (5.1) for all considered examples (aside from other terms) an additional term of the form $\propto u$, which breaks Galilean invariance and also changes the dynamic behavior substantially [79]. Accordingly so far no observation of a second dynamic degree of freedom has been reported for the examples studied leading to the Sivashinsky equation.

In [54] Coullet and Fauve discuss in all generality the potential impact of Galilean invariance on phase dynamics. In [55] the same authors propose that in a one-dimensional, spatially periodic structure with a mean flow a phase dynamic equation arises, which is second order in time, because of Galilean invariance.
Coullet and Fauve start from the amplitude equations

$$A_t = A + A_{xx} - |A|^2 A - iq_0\, A\, B - B\, A_x \qquad (5.4)$$

$$B_t = \lambda\, B_{xx} + |A|^2_x - B\, B_x \qquad (5.5)$$

and then show, that a nonlinear phase dynamics results, which is second order in time and gives rise to a propagating excitation. An application of (5.4) and (5.5) to convection is not possible, however. Whatever scaling is used for B, ∂_t and ∂_x, one never obtains the amplitude equations (5.4) and (5.5). Especially (5.5) is not dynamic, if one choses, as an example, the scaling with the distance from onset familiar from Rayleigh Benard convection in simple fluids as discussed by Newell and Whitehead [17].This can be read off immediately from eq.(1b) of the paper by Siggia and Zippelius [59] , keeping in mind that their coefficient γ depends on ε. This might be different in the limit of very small Prandtl numbers [55,60], a case discussed especially by Coullet and Huerre [60]. This limit clearly deserves further investigation. We would like to emphasize, however, that even for free slip boundary conditions there is no scaling for the Prandtl number and the distance from onset, which would render both, the amplitude equation and the equation for the vertical vorticity dynamic in lowest consistent order in an expansion with the distance from onset.

The influence of the vertical vorticity on the phase dynamics of convective systems

As has been shown by Cross [61], the incorporation of vertical vorticity into the phase dynamics for stationary convection in the Rayleigh Benard instability leads to a modification of the phase diffusion equation for finite Prandtl number

$$\phi_t = D_{||}\partial_{xx}\phi + D_{\perp}\partial_{yy}\phi + U \qquad (6.1)$$

where U is a drift term, which comes from the horizontal velocity, which is in turn driven by phase gradients. If one eliminates U one ends up with an equation for the phase only. The price one has to pay in some cases is a singular expansion in the gradients [61] (cf., however, the discussion by Manneville and Piquemal [80] for the zig-zag instability). It seems important to note, however, that no additional dynamic degree of freedom is arising by the incorporation of the mean drift. This is also clear from inspection of the amplitude equations derived by Siggia and Zippelius [59]: in lowest consistent order the equation for the mean flow is static.

This situation changes, when going to the oscillatory instability with traveling waves, as they arise e.g. near the onset of convection in binary fluid mixtures. In this case we obtain from the amplitude equations derived by Brand,Lomdahl and Newell [30,31]

$$\phi_t = N(\phi) - \varepsilon k_c U \qquad (6.2)$$

$$-P^{-1}\partial_t U + U_{yy} = 4(Pk_c)^{-1} r^2 \phi_y \qquad (6.3)$$

where $N(\phi)$ is a generalization of the expression $\phi_{xx} + \phi_{xxxx} + (\phi_x)^2$ in the Kuramoto equation discussed above.
Thus one finds for an oscillatory instability with travelling waves and slow spatial variations in both directions of the plane for systems with broken rotational symmetry two dynamic degrees of freedom in phase dynamics. A detailed discussion of (6.2) and (6.3) along with the explicit expression of $N(\phi)$ can be found elsewhere [62].

Phase dynamics of pattern forming equilibrium systems in an external field oder under an external load

All examples discussed so far have addressed pattern forming nonequilibrium systems including convective rolls and Taylor vortices.

There are also equilibrium systems, however, which form static spatial patterns under the influence of external fields or external forces. As examples we mention: the Rosenzweig instability in ferrofluids [63], the dimple instability arising for electrons at helium interfaces [64] and the deformations of an elastic plate (buckling) under an external load [65]. These systems are intermediate between pattern forming nonequilibrium systems on one hand and systems close to local thermodynamic equilibrium without macroscopic spatial patterns on the other.
Therefore it seems natural to investigate to what extent the concept of phase dynamics developed for dissipative pattern forming nonequilibrium systems can be carried over to pattern forming equilibrium systems. [illegible]uestion has been addressed very recently by Brand and Wesfrei[illegible]] and we will only summarize here some of their main conclusions.
In contrast to dissipative nonequilibrium systems, which give typically rise to phase equations, which are first order in time, equilibrium systems with an external constraint possess a kinetic term in the generalized free energy.
If we consider a system with a one-dimensional spatially periodic structure - as it is e.g. approximately the case for a long, rectangular elastic plate under the influence of an external load - we obtain for the generalized free energy asociated with changes of the wavelength [66]

$$F \propto \int \alpha\, \phi_t^2 + \beta\, \phi_x^2 \qquad (7.1)$$

where x is the direction perpendicular to the unit cells (e.g. rolls) and where we have assumed $x \to -x$ symmetry, as it is satisfied for 'buckling' in the configuration described above. The coefficient α corresponds to the mass density. From (7.1) we obtain the dynamic equation

$$\alpha\, \phi_{tt} - \beta\, \phi_{xx} = 0 \qquad (7.2)$$

and for $\beta < 0$ keeping a higher order gradient term

$$\alpha \phi_{tt} - \beta \phi_{xx} - \gamma \phi_{xxxx} = 0 \qquad (7.3)$$

For $\beta < 0$ the associated instability of the phase is the Eckhaus instability against compression and dilatation of the periodic structure.
For the dispersion relation characterizing low frequency, long wavelength excitations we obtain [66]

$$\omega^2 = \beta/\alpha\, k^2 + \gamma/\alpha\, k^4 \qquad (7.4)$$

i.e. a propagating mode with velocity $c^2 = \beta/\alpha$. Dissipation and nonlinearities can be incorporated along the lines indicated above [66].
A generalization to the case of hexagons, which arise quite frequently in pattern forming equilibrium system under an external constraint (e.g. for the Rosenzweig instability in ferrofluids, for the dimple instability of electrons at helium interfaces and for the buckling of thin shells) has also been examined [66] and leads to two pairs of propagating modes. This result turns out to be true as well for the less frequently observed square patterns.
In conclusion we would like to stress, that also for the case of pattern forming equilibrium systems under external constraints a description can be given, which is valid on length scales large compared to the period of the underlying spatial pattern and thus on length scales much larger than those of classical hydrodynamics. I.e. also for the systems described in this section the sequence: Newton's law, Boltzmann equations, hydrodynamic equations, can be extended by phase dynamics.

Phase dynamics with a material derivative due to a flow field

In the last section we have seen that phase dynamics is not only applicable to large aspect ratio pattern forming nonequilibrium systems, but that it can also be used to describe spatial patterns close to equilibrium in systems with an external constraint.

In this section we extend our previous investigations in a different direction. We address the question to what extent an internally or externally imposed flow field changes the phase dynamic equations discussed so far. This issue has been considered in detail in a recent paper [67] and so we can be brief here.

There are at least three experimental observations [6,68-70], which indicate the importance of this question. Ref.68 describes the effect of a Poiseuille flow through a rectangular container, which is studied near the onset of Rayleigh Benard convection. Luijkx et al. [68] report a propagation of the roll pattern with apparently no change in wave vector. Wimmer [69,70] investigated the flow between concentric rotating cones and found a propagation of the vortices as well as a variation of the pair size along the generating line of the cones. Quite recently, Pocheau et al. [6] reported for a well controlled,quantitative convection experiment in an annulus with a through flux, a change in wavenumber of the rolls along the annulus and no propagation of the rolls.

These experimental results naturally call for a theoretical description. We start with the analysis of [6]. Clearly the external flux breaks the $x \to -x$ symmetry, where x denotes the coordinate along the azimuthal angle. Since a stationary situation is observed, a driving term proportional to the external flow field v is ruled out. The lowest order nontrivial term satisfying all requirements listed is of the form $v\partial_x\phi$ where the phase ϕ denotes the position of the rolls along the annulus.

This consideration leads naturally to a phase equation of the form

$$\partial_t\phi + v\partial_x\phi = D_{||}\phi_{xx} \qquad (8.1)$$

which describes the transformation of the phase diffusion equation (2.1) to a situation with advection due to the imposed flow. There is no reason, however, to write down an additional dynamic equation for v, since the flux is completely externally controlled.

For a stationary situation as observed in [6] one obtains from (8.1) for the change Δk in the wavevector k as one moves along the cell

$$\Delta k/k \approx v/D_{||} \qquad (8.2)$$

Equation (8.2) has been confirmed quantitatively near onset of convection in [6] and we refer to their thorough discussion for the details (cf. also the contribution by Pocheau [71] to this volume).
The generalization of (8.1) to the nonlinear terms including higher order spatial derivatives follows the procedure outlined above for other cases and we refer to [67] for the details.
Near threshold an amplitude equation of the form

$$A_t + v\partial_x A = \varepsilon A + D_{||}\ A_{xx} - g\,|A|^2 A \qquad (8.3)$$

gives rise to (8.1), as is easily shown using the technique developed by Newell and coll. [17,27-31].
For the flow between concentric rotating cones clearly a driving as well as a convective term in the phase equation are necessary to account for both, the propagation of the vortices and the variation of the pair size observed by Wimmer [69,70]. At the conference reported here, I learned from Pocheau [71], that a similar situation also arises for the case of convection in an annulus with an imposed flow field at higher flux rates.
This leads to suggest the following minimal model phase equation to describe the results by Wimmer and by Pocheau et al.

$$\partial_t\phi + v\partial_x\phi = D_{||}\,\phi_{xx} + \alpha\, v \qquad (8.4)$$

From inspection of (8.4) it is clear that the convective term and the diffusion can be combined to give a variation in unit cell size whereas the time derivative and the driving term yield a propagation.of the pattern. An amplitude equation similar to (8.3) is easily derived for the present case and it turns out that one only needs to add the term iAv to the right hand side of (8.3) to account for the propagation effect.
Clearly (8.4) should be tested in detail to make sure that other contributions are not relevant to account for the experimental results. The results reported in [68] seem to indicate that there are also situations where there is only propagation and no variation of

the roll size. This observation is also contained in (8.4) as a special case.

Thus we arrive at the conclusion that there are four possible cases
a) there is no external or internal flow and the phase diffusion equation suggested by Pomeau and Manneville applies
b) an external flow can give rise to a change in roll size without a propagation of the pattern
c) there is a propagation without change in the size distribution and
d) both, a variation in wavelength and a propagation occur.

And all four cases have been verified experimentally [6,15,16,68-71].

As a next step, which still needs to be done, it seems now important to predict from the basic equations which one of the three scenarios b)-d) arises for a given set of parameters in each experimentally relevant situation.

We close this section by pointing out, that in none of the cases considered, it was necessary to write down a dynamic equation for the flow field (external or internal). This fact is nicely complemented by the observation of the mean flow close to the onset of convection in a circular container [7] by Croquette et al., for which it has also been unnecessary to invoke an additional dynamic degree of freedom to account for the results. Thus the additional terms discussed above seem to be sufficient to cover all the reported results near the onset of convection in simple fluids, even in the presence of mean flow or external flow fields

Conclusions and Perspective

In the bulk part of the present survey we have discussed the phase dynamics for a number of large aspect ratio pattern-forming nonequilibrium systems, concentrating on their excitations and on the question how the phenomenological coefficients in these phase equations can be related to more conventional approaches such as amplitude equations. The most important point, which needs to be checked experimentally to establish phase dynamics as the analogue of hydrodynamics for large aspect ratio pattern forming nonequilibrium systems, is the occurrence of propagating modes. A number of systems which might show such an excitation has been suggested here.

Other questions, which are closely related to phase dynamics and have not been reviewed here, include the dynamics of defects in nonequilibrium patterns and the field of pattern selection mechanisms. For the former problem two approaches have been put forward in 1984. Cross and Newell [72] extract the motion of defects from their nonlinear phase dynamics for convective patterns. The second approach has been suggested by Kawasaki and the present author [73-75]. Here the idea is to derive a kinematic equation for each defect for a small number of defects and to couple the resulting equations to phase dynamics. More recently Coullet et al [48,49] have followed up with yet another way of looking at the problem. For neither of the three points of view there has been a critical experimental test. It therefore seems a major task for each approach to come up with suggestions for experiments which could be used to distinguish between the theoretical ways of looking at the problem.

Another important issue is the question of pattern selection or wavelength selection in large aspect ratio pattern forming nonequilibrium systems. Loosely speaking the problem arising is how to guarantee that e.g. a Benard cell contains a certain number of rolls and not just any number within the band of wavevectors allowed by the Eckhaus instability. In 1982 a partial solution to this problem was given by Kramer, Benjacob, Brand and Cross [76], who used a phase dynamic type approach to show that for a sufficiently smooth change of the bifurcation parameter as a function of the spatial coordinate a unique wave number is selected. In this paper the example chosen was a system of reaction diffusion equations. Shortly thereafter this prediction was tested experimentally for the vortex flow state of the Taylor instability by Ahlers, Cannell and Dominguez-Lerma [77]. They designed a Taylor cell in which one of the cylinders had over part of its length a wedge shaped geometry. Selection was found and since then the mechanism was used to explore further details of pattern selection [10,78].

In closing we would like to emphasize that only the analogues of broken translational symmetry and of combined broken translational and rotational symmetry have been found in phase dynamics. Neither broken rotational symmetry nor broken gauge invariance, which is characteristic of superfluids, have as yet found an analogue in large aspect-ratio pattern forming nonequilibrium systems. A nonequilibrium analogue of broken rotational symmetry could possibly be found in disordered convective patterns in a circular cell [12], in which one has patches of rolls or hexagons (if up-down symmetry is violated) with a given orientation. If rotational symmetry in such a system is broken, it could be detected experimentally by the presence of a diffusive mode. So far, however, there is no evidence for this behavior, nor has it been suggested before theoretically.

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